Applied Partial Differential Equations
Equations aux dérivées partielles appliquées

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MO'TASSEM AL-ARYDAH, Ottawa

DAVID AMUNDSEN, Carleton University

Standing waves in a partitioned tube with passive membrane

The propagation of waves within a tube containing disparate gases separated by a passive membrane is modeled and analyzed in the limit of weak dissipation and applied forcing. This provides a simple setting in which to study the nonlinear interactions within and between each gas and provides a paradigm for other similar physical systems such as laminated elastic materials. The associated resonant frequencies are found in terms of a linear functional equation involving a non-trivial combination of the separate natural frequencies. As expected, in the limit that the gases have the same material properties, the modes become commensurate and the model reduces to that of the classical shock tube. However, sufficiently away from this limit it is seen that this structure is lost and smooth single mode resonant solutions arise.

Using a perturbative approach these solutions are approximated and compared to numerical solutions of the full system. The transition between smooth and discontinuous solutions is also studied both numerically and analytically, based on a dimensionless parameter associated with the relative material difference.

YVES BOURGAULT, University of Ottawa

Simulation of flow and aerosol in the airway tract

The numerical prediction of the propagation of aerosols in airways is an area of growing interest. For instance, predicting the deposition patterns of particles in the mouth-throat or the inner lung pathways is needed for designing inhalers. Two approaches are commonly used to model such multiphase flows, namely the Eulerian and Lagrangian approaches, but much of the literature on aerosols in airways rely on the Lagrangian tracking approach. In our talk, we will show that an Eulerian model can efficiently predict aerosol propagation in a 3-D patient-based geometry of the airway tract. CT images of the thorax were processed to generate the geometry of the trachea and the first four bronchus generations. The air flow was then obtained by solving the Navier–Stokes equations for different values of the flow dimensionless parameters. Numerical results for aerosol propagation in these geometries were obtained by coupling our Eulerian model with the previous flow solutions. The results show that the particle density and deposition patterns are easily obtained at all time steps without any need for particle count or the delicate selection of initial particle positions, as for the Lagrangian tracking approach. The general simulation strategy, as well as techniques to handle critical aspects of the simulations, will be covered in the talk.

JOHN BOWMAN, University of Alberta

A Fully Lagrangian Advection Scheme

A numerical method for passive scalar and self-advection dynamics, *Lagrangian rearrangement*, is proposed. This fully Lagrangian advection algorithm introduces no artificial numerical dissipation or interpolation of parcel values. In the inviscid limit, it preserves the infinity of Casimir invariants associated with parcel rearrangement. In the two-dimensional case presented here, these invariants are arbitrary C^1 functions of the vorticity and concentration fields. The initial parcel centroids are evolved in a Lagrangian frame, using the method of characteristics. At any time this Lagrangian solution may be viewed by projecting it onto an Eulerian grid using a rearrangement map. The resulting rearrangement of initial parcel values is accomplished with a weighted Bresenham algorithm, which identifies quasi-optimal, distributed paths along which chains of parcels are pushed to fill in nearby empty cells. The error introduced by this rearrangement does not propagate to future time steps.

LUCY CAMPBELL, Carleton University, 1125 Colonel By Drive, Ottawa, Ontario, K1S 5B6
Rossby wave propagation in a zonally-varying basic flow

Theoretical studies of the propagation of barotropic Rossby waves are generally based on the assumption that the basic flow is a function of latitude only, with no variation in the longitudinal (zonal) direction. In this configuration the linear and nonlinear dynamics of Rossby waves have been studied extensively in the past few decades, with a great deal of focus being on the interactions of the waves with the basic flow at critical latitudes. It is well known that the "surf zone" around a critical latitude acts as a barrier to the propagation of Rossby waves from the mid-latitudes to the tropics. This would lead one to expect the equatorial region to develop independently of the mid-latitude zones, at least as far as wave dynamics and momentum and energy transport are concerned. However, analyses of upper tropospheric data have found evidence of zonal variations in the time-averaged zonal wind, as well dynamic links between the mid-latitudes and the tropics, which would suggest that there is some degree of equatorward transmission of transient waves from the mid-latitudes. A possible explanation for these observations is that the basic flow varies in the zonal direction. We shall examine these issues in the context of a beta-plane configuration in which the basic flow velocity has a zonally-varying component. The mathematical problem, as we define it, is governed by three small parameters, which define the amplitude of the waves relative to the basic flow, the magnitude of the zonally-varying component of the basic flow relative to the zonally-uniform part, and the zonal extent of the zonally-varying component. Linear analyses and nonlinear numerical simulations will be presented.

This is joint work with former MSc student Mariam Tariri.

ALEXEI CHEVIAKOV, University of Saskatchewan
Analytical properties and exact solutions of anisotropic MHD equilibrium systems in three dimensions

For static reductions of isotropic and anisotropic magnetohydrodynamics (MHD) plasma equilibrium models, a complete classification of point symmetries and conservation laws up to first order is presented. It is shown that the symmetry algebra for the isotropic equations is finite-dimensional, whereas anisotropic equations admit infinite symmetries depending on a free function defined on the set of magnetic surfaces.

A direct transformation is established between isotropic and anisotropic equations, which provides an efficient way of constructing new exact anisotropic solutions. In particular, axially and helically symmetric anisotropic plasma equilibria arise from classical Grad–Shafranov and JFKO equations.

This is a joint work with S. Anco (Brock).

PAUL CHOBOTER, Cal Poly State University, San Luis Obispo, California
Dynamics and internal structure of the cross-shelf circulation during wind-driven coastal upwelling

A two-dimensional theory of wind-driven coastal upwelling is developed that is comprised of a surface Ekman layer, an interior frictionless layer, and a frictional bottom boundary layer. The theory is built upon the Lentz–Chapman upwelling theory, which has been used to demonstrate the importance of nonlinear cross-shelf momentum flux divergence during upwelling. The new model adds to the Lentz–Chapman theory a spatially-varying structure in the interior density and velocity fields. The dynamical model for the interior flow is based upon the nonlinear upwelling theory of Pedlosky, which maintains thermal wind balance between the cross-shelf density gradient and the vertical shear in the alongshelf velocity while retaining the cross-shelf advection of density and alongshelf momentum. The structure of the cross-shelf circulation is studied as a function of Burger number $S = \alpha N/f$, where α is the topographic slope, N is the buoyancy frequency, and f is the Coriolis parameter. Predictions of the dynamical model are compared with numerical simulations using the Regional Ocean Modeling System (ROMS, a primitive-equation community model). During upwelling winds, the dynamical model predicts interior onshore flow high in the water column for large Burger number, and onshore flow in the bottom boundary layer for small Burger number, consistent with the numerical simulations and with observations.

SERGE D'ALESSIO, University of Waterloo, Dept. of Applied Math, 200 University Ave. West, Waterloo, ON, N2L 3G1
Instability of Gravity-Driven Flow Over Uneven Surfaces

Discussed in this talk is the gravity-driven two-dimensional laminar flow of a thin layer of fluid down an uneven incline. In particular, the effect of bottom topography and surface tension on the stability of the flow was investigated. The equations of motion are non-hydrostatic approximations to the Navier–Stokes equations which exploit the assumed relative shallowness of the fluid layer. The explicit dependence on the cross-stream coordinate is eliminated from the equations of motion by means of a weighted residual approach. The resulting mathematical formulation constitutes an extension of the modified integral-boundary-layer equations first proposed by Ruyer–Quil and Manneville (see Eur. Phys. J. B **15**(2000), pp. 357–369) for flows over even surfaces to flows over variable topography. A linear stability analysis of the steady flow is carried out using Floquet–Bloch theory. Results from the linear analysis are used to initiate a weakly nonlinear analysis based on asymptotic theory. A numerical solution procedure is also used to solve the nonlinear governing equations and to calculate the evolution of the perturbed equilibrium flow. The numerical simulations serve to confirm the analytical predictions and to investigate the interfacial wave structure. The bottom topography considered in this study corresponds to a sinusoidal profile characterized by a wavelength and amplitude. Conclusions are drawn on the combined effect of bottom topography and surface tension.

MICHAEL HASLAM, York University
Three-Dimensional Curved Wire Antenna Problems

Partial differential equations are often reformulated as surface integral equations, via, say, Green's theorem. Such a reformulation can be highly advantageous from a numerical point of view, since it makes the use of domain-termination techniques unnecessary and can be used in conjunction with high-order numerical integration schemes. In this talk I discuss our recent work on fast, high-order algorithms applied to the thin wire antenna problem. Based on the electric field integral equation, we develop a model applicable to three-dimensional curved wires. The algorithms we propose converge super-algebraically: faster than $\mathcal{O}(1/N^m)$ and $\mathcal{O}(1/M^m)$ for any positive integer m, where N and M are the numbers of unknowns and the number of integration points required for construction of the discretized integral operator, respectively. Previous methods are limited to low order convergence rates due to a tangent line approximation used to resolve the singularity of integral operator. We illustrate the effectiveness of our methods with applications to wires described by both closed and open curves; the latter case requires some special treatment of end-point singularities.

With Oscar Bruno, Caltech.

NICHOLAS KEVLAHAN, McMaster University
Vortices for computing: the engines of turbulence simulation

Vortices have been described as the "sinews of turbulence" (Moffatt et al., 1994). They are also, increasingly, the computational engines driving numerical simulations of turbulence. In this talk I describe some recent advances in vortex-based numerical methods for simulating high Reynolds number turbulent flows. I focus on coherent vortex simulation (CVS), where nonlinear wavelet filtering is used to identify and track the few high energy multiscale vortices that dominate the flow dynamics. This filtering drastically reduces the computational complexity for high Reynolds number simulations. It also has the advantage of decomposing the flow into two physically important components: coherent vortices and background noise. In addition to its computational efficiency, this decomposition provides new insight into the structure and dynamics of high Reynolds number turbulence.

BOUALEM KHOUIDER, University of Victoria
A stochastic multicloud model for organized tropical convection

We propose a stochastic model for representing the missing variability in global climate models due to unresolved features of organized tropical convection. We use a Markov chain lattice model to represent small scale convective elements which interact with each other and with the large scale environmental variables through convective available potential energy (CAPE) and middle troposphere dryness. Each lattice site is either occupied by a cloud of a certain type (congestus, deep or stratiform) or it is a clear sky site. The lattice sites are assumed to be independent from each other so that a coarse-grained stochastic birth-death system, which can be evolved with a very low computational overhead, is obtained for the cloud area fractions alone. The stochastic multicloud model is then coupled to a simple tropical climate model consisting of a system of ode's, mimicking the dynamics over a single GCM grid box. Physical intuition and observations are employed here to constrain the design of the models. Numerical simulations showcasing some of the dynamical features of the coupled model are presented below.

Joint work with J. Biello and A. Majda.

GREG LEWIS, UOIT

Flow transitions in a differentially heated rotating channel of fluid

We study the primary flow transitions that occur in a differentially heated rotating channel of fluid by computing bifurcations in a model that uses the Navier–Stokes equations in the Boussinesq approximation. When the centrifugal buoyancy is neglected, the system is $O(2)$-symmetric. In this case, the flow transition corresponds to a symmetry-breaking steady-state bifurcation. We use numerical continuation to trace the transition over a wide range of the two parameters of interest. At isolated points along the transition curve, centre manifold reduction and normal forms are used to deduce the form of the bifurcating solutions. The solutions and transitions are approximated numerically from the large sparse systems that result from the discretization of the partial differential model equations.

The channel can be considered to be a simplification of the classical differentially heated rotating annulus fluid dynamics experiment. The results obtained for the channel show a remarkable quantitative correspondence with both experimental and theoretical results from the annulus.

Joint with Matthew Hennessy, UOIT.

EMMANUEL LORIN, University of Ontario Institute of Technology

Study of a micro-macro model for electromagnetic field propagation in a gas

In this work we are interested in the study of a micro-macro Maxwell–Schroedinger system, modeling the propagation of intense and high frequency electromagnetic fields in dense gaseous media (Comput. Phys. Comm. **177**, 2007; New J. Physics **10**, 2008). In the first part of the talk, we will focus on the existence and uniqueness of weak solutions, and some properties of the model related to ionization, plasma effects and high order nonlinearities. In the second part, we are interested in some particular but fundamental (for applications) solutions called filaments. In this goal, we will propose some numerical simulations, and a formal comparison of the Maxwell–Schroedinger system with some existing nonlinear wave and Schroedinger equations, that are well known to possess filaments as solutions.

This work is a collaboration with Prof. A. Bandrauk, and Dr. S. Chelkowski (University of Sherbrooke). Simulations are performed on the HPC Mammouth (RQCHP).

MING MEI, Champlain College and McGill University

Diffusion Waves for Hyperbolic p-System with Nonlinear Damping

This talk is concerned with the p-system of hyperbolic conservation laws with nonlinear damping. We show that the solutions of the Cauchy problem for the p-system converge to their corresponding nonlinear diffusion waves, which are the solutions of the corresponding nonlinear parabolic equation given by the Darcy's law. The optimal convergence rates are also obtained. In order to overcome the difficulty caused by the nonlinear damping, a couple of correction functions have been technically constructed. The approach adopted is the elementary energy and Fourier transform.

ABBAS MOMENI, Queen's University, Kingston, ON
A variational principle associated to a certain class of boundary value problems

A variational principle is introduced to provide a new formulation and resolution for several boundary value problems. Indeed, we consider systems of the form

$$\begin{cases} \Lambda u = \nabla \Phi(u), \\ \beta_2 u = \nabla \Psi(\beta_1 u) \end{cases}$$

where Φ and Ψ are two convex functions and Λ is a possibly unbounded self-adjoint operator modulo the boundary operator $\mathcal{B} = (\beta_1, \beta_2)$. We shall show that solutions of the above system coincide with critical points of the functional

$$I(u) = \Phi^*(\Lambda u) - \Phi(u) + \Psi^*(\beta_2 u) - \Psi(\beta_1 u)$$

where Φ^* and Ψ^* are Fenchel–Legendre dual of Φ and Ψ respectively. Note that the standard Euler–Lagrange functional corresponding to the system above is of the form,

$$F(u) = \frac{1}{2}\langle \Lambda u, u\rangle - \Phi(u) - \Psi(\beta_1 u).$$

An immediate advantage of using the functional I instead of F is to obtain more regular solutions and also the flexibility to handle boundary value problems with nonlinear boundary conditions. Applications to Hamiltonian systems and semi-linear elliptic equations with various linear and nonlinear boundary conditions are also provided.

CHUN HUA OU, Memorial University of Newfoundland
How Many Consumer Levels Can Survive in a Chemotactic Food Chain?

We investigate the effect and the impact of predator-prey interactions, diffusivity and chemotaxis on the ability of survival of multiple consumer levels in a predator-prey microbial food chain. We aim at answering the question of how many consumer levels can survive. To solve this standing issue on food-chain length, first we construct a chemotactic food chain model. A priori bounds of the steady state populations are obtained. Then under certain sufficient conditions combing the effect of conversion efficiency, diffusivity and chemotaxis parameters, we derive the co-survival of all consumer levels, thus obtaining the food chain length of our model. Numerical simulations not only convince our theoretical results, but also demonstrate the impact of conversion efficiency, diffusivity and chemotaxis behavior on the survival and stability of various consumer levels.

FRANCIS POULIN, University of Waterloo
Magneto-Shallow Water Waves

It is believed that stars have a thin layer called the tachocline that exists below the convection zone. The thinness of this layer allows for the shallow water approximation to be applied to the magneto-hydrodynamic equations. This yields the Magneto-Shallow Water equations. In this simplified model there are various waves that can co-exist, namely; Poincaré and Rossby waves that are modified by the magnetic field, as well as Alfvén waves that are modified by the hydrodynamics.

In previous work in the literature, analytical expressions were obtained to approximate the dispersion relation of these different waves in both Cartesian and spherical coordinates. This work extends the results in the literature by dropping the approximations necessary to derive analytical solutions and computing these solutions using a spectrally accurate Chebyshev collocation method. The waves that we compute need not be harmonic and in general are trapped. We consider the case where the magnetic field varies sinusoidally with latitude but it is very easy to adapt our method to any field that is longitudinally invariant.

BARTOSZ PROTAS, Department of Mathematics & Statistics, McMaster University, Hamilton, ON
An Inverse Formulation for Solution of Free-Boundary Problems

In this presentation we are concerned with the two-phase solidification (Stefan) problem as a common example of a free-boundary problem. We reformulate this problem as an inverse problem in which a cost functional is minimized with respect to the position of the interface and subject to suitable PDE constraints. An advantage of this formulation is that it allows for a thermodynamically consistent treatment of the interface conditions in the presence of a contact point involving a third phase. Furthermore, such a formulation also makes it possible to solve optimal control problems in essentially the same way as the "direct" problem. We describe an efficient iterative solution method for the inverse formulation of the Stefan problem which uses shape differentiation and adjoint equations to determine the gradient of the cost functional. Performance of the proposed approach is illustrated with sample computations concerning 2D steady solidification phenomena. This investigation is a part of a broader research effort related to optimization of advanced welding techniques used in automotive manufacturing.

Joint work with Oleg Volkov.

MAREK STASTNA, University of Waterloo, 200 University Ave. West, Waterloo, ON, N2L 3G1
Is mathematical complexity worth the price? Two case studies from stratified fluid dynamics

Fluid dynamics has had a long-standing and profound influence on the mathematics of partial differential equations. In this talk I will present two case studies from the dynamics of stratified fluids. The first case study will consider the Dubreil–Jacotin–Long equation for fully nonlinear, internal solitary waves. I will demonstrate that the price of a more complex mathematical description yields several novel physical predictions, and that these predictions are missed by simpler, weakly nonlinear theories. The second case study will consider the so-called nontraditional component of the Coriolis force in a rotating frame of reference. I will show that the introduction of this term alters classical normal modes for a basin (a lake, for example) and that Kelvin waves are formally no longer possible. However, as numerical simulations indicate, this point is largely irrelevant for time scales on the order of days. I will contrast the two case studies and speculate on some general implications of the results.

GORDON SWATERS, University of Alberta, Department of Mathematical and Statistical Sciences, Edmonton, Alberta, T6G 2G1
Mixed frictional-Kelvin–Helmholtz destabilization of source-driven abyssal overflows in the ocean

Source-driven ocean currents that flow over topographic sills are important initiation sites for the abyssal component of the thermohaline circulation. These overflows exhibit vigorous space and time variability over many scales as they progress from a predominately gravity-driven down slope flow to a geostrophic along slope current. Observations show that in the immediate vicinity of a sill, grounded abyssal ocean overflows can possess current speeds greater than the local long internal gravity wave speed with bottom friction and down slope gravitational acceleration dominating the flow evolution. It is shown that these dynamics lead to the mixed frictionally-induced and Kelvin–Helmholtz instability of grounded abyssal overflows. Within the overflow, the linearized instabilities correspond to bottom-intensified baroclinic roll waves and in the overlying water column amplifying internal gravity waves are generated. The stability characteristics are described as a function of the bottom drag coefficient and slope, Froude, bulk Richardson and Reynolds numbers associated with the overflow and the fractional thickness of the abyssal current compared to the mean depth of the overlying water column. The marginal stability boundary and the boundary separating the parameter regimes where the most unstable mode has a finite or infinite wavenumber are determined. When it exists, the high wavenumber cut-off is obtained. Conditions for the possible development of an ultra-violet catastrophe are determined. In the infinite Reynolds number limit, an exact solution is obtained which fully includes the effects of mean depth variations in the overlying water column associated with a sloping bottom. For parameter values characteristic of the Denmark Strait overflow, the most unstable mode has wavelength of about 19 km, a geostationary period of about 14 hours, an e-folding amplification time of about 2 hours and a down slope phase speed of about 74 cm/s.

MICHAEL WAITE, University of Victoria, PO Box 3060 Stn CSC, Victoria, BC, V8W 3R4
Investigating the atmospheric kinetic energy spectrum with numerical simulation

Numerical simulation provides a powerful tool in the effort to understand the atmospheric kinetic energy spectrum. Experiments typically fall in to one of two categories:

(1) idealized simulations of rotating stratified turbulence, usually with a pseudo-spectral integration of the Boussinesq equations; or

(2) simulations with a global or mesoscale atmospheric model, in which a range of physical effects (e.g., clouds, radiation) are represented in some fashion.

In this talk, we will present new high-resolution simulations of an atmospheric baroclinic wave, and discuss the implications for theories of the kinetic energy spectrum. Forcing from clouds, radiation, and topography are ignored as in (1), but a realistic atmospheric flow is employed as in (2). The tropospheric spectrum is steeper than observed through the mesoscale, although the horizontally divergent contribution has a realistic slope of around $-5/3$. These results raise questions about the applicability of turbulence cascade theories to the atmospheric spectrum.