Partial Differential Equations
Équations aux dérivées partielles
(Org: **Henri Berestycki** (Paris) and/et **Robert Jerrard** (Toronto))

LIA BRONSARD, McMaster University, Hamilton, Ontario, Canada
Global minimizers for anisotropic superconductors / Sur les minimiseurs globaux des supraconducteurs anisotropes

We study periodic minimizers of the Anisotropic Ginzburg–Landau and Lawrence–Doniach models for anisotropic superconductors, in various limiting regimes. We are particularly interested in determining the direction of the internal magnetic field (and vortex lattice) as a function of the applied external magnetic strength and its orientation with respect to the axes of anisotropy. We identify the corresponding lower critical fields, and compare the Lawrence–Doniach and anisotropic Ginzburg–Landau minimizers in the periodic setting.

This talk represents joint work with S. Alama and E. Sandier.

Nous étudions les minimiseurs périodiques du modèle de Ginzburg–Landau anisotrope ainsi que ceux du modèle de Lawrence–Doniach pour les supraconducteurs de hautes-températures anisotropes, et ce pour plusieurs régimes asymptotiques. Nous déterminons la direction du champs magnétique induit (et donc du réseau de vortex,) comme fonction de l'amplitude et de la direction du champs magnétique imposé. En particulier, nous trouvons le premier champs critique, et comparons les minimiseurs des modèles de Lawrence–Doniach et de Ginzburg–Landau anisotrope.

Ceci est un travail commun avec Stan Alama et Etienne Sandier.

JAMES COLLIANDER, Toronto

WALTER CRAIG, McMaster University, Department of Mathematics & Statistics
Navier–Stokes equations and their Kolmogorov spectra

Suppose that $u(x,t)$ is a (possibly weak) solution of the Navier–Stokes equations on all of $\mathbb{R}^3$, or on the torus $\mathbb{R}^3/\mathbb{Z}^3$. Let $\hat{u}(k,t)$ denote its Fourier transform. Then the *energy spectrum* of $u(\cdot,t)$ is the spherical integral

$$E(\kappa,t) = \int_{|k|=\kappa} |\hat{u}(k,t)|^2 \, dS(k), \quad 0 \le \kappa < \infty,$$

or alternatively, a suitable approximate sum. An argument invoking scale invariance and dimensional analysis given by Kolmogorov in 1941, and subsequently refined by Obukov, predicts that large Reynolds number solutions of the Navier–Stokes equations in three dimensions should obey

$$E(\kappa,t) \sim C\kappa^{-5/3},$$

at least in an average sense and over some of the range of κ. I will describe a global estimate on weak solutions in the norm $|\mathcal{F}\partial_x u(\cdot,t)|_\infty$ which gives bounds on a solution's ability to satisfy the Kolmogorov law. The result gives rigorous upper and lower bounds on the inertial range, and an upper bound on the time of validity of the Kolmogorov spectral regime.

This is joint work with Andrei Biryuk.

MARIA ESTEBAN, CNRS et Université Paris–Dauphine
Relation between Onofri and Caffarelli–Kohn–Nirenberg inequalities

I will first discuss a class of inequalities of Onofri type depending on a parameter, in the two-dimensional Euclidean space.

Then I will explain how these inequalities give us some insight on the symmetry breaking phenomenon for the extremal functions of the Caffarelli–Kohn–Nirenberg inequality, in two space dimensions. In fact, for suitable sets of parameters (asymptotically sharp) we prove symmetry or symmetry breaking by means of a blow-up method and a careful analysis of the convergence to a solution of a Liouville equation. In this way, the Onofri inequality appears as a limit case of the Caffarelli–Kohn–Nirenberg inequality.

This is joint work with J. Dolbeault and G. Tarantello

NASSIF GHOUSSOUB, University of British Columbia

Estimates for the quenching time of a parabolic equation modeling electrostatic MEMS

We analyze the nonlinear parabolic problem $u_t = \Delta u - \frac{\lambda f(x)}{(1+u)^2}$ on a bounded domain Ω of $\mathbf{R}^N$ with Dirichlet boundary conditions. This equation models a simple electrostatic Micro-Electromechanical System (MEMS) consisting of a thin dielectric elastic membrane with boundary supported at 0 above a rigid ground plate located at -1. When a voltage—represented here by λ—is applied, the membrane deflects towards the ground plate and a snap-through may occur at finite "quenching time" $T_\lambda(\Omega, f)$ when it exceeds a certain critical value $\lambda^*(\Omega, f)$ (pull-in voltage). This creates a so-called "pull-in instability" which greatly affects the design of many devices. The challenge is to estimate $\lambda^*(\Omega, f)$ and $T_\lambda(\Omega, f)$ in terms of the shape (geometry) and the material properties of the membrane, which can be fabricated with a spatially varying dielectric permittivity profile f.

This is a joint work with Yujin Guo.

EMMANUEL GRENIER, ENS, Lyon

STEPHEN GUSTAFSON, University of British Columbia, Mathematics Dept., 1984 Mathematics Rd., Vancouver, BC, Canada V6T 1Z2

Scattering theory for the Gross–Pitaevskii equation in three dimensions

For the Gross–Pitaevskii (nonlinear Schroedinger) equation, which models superfluids (among other things), it is natural to consider non-zero boundary conditions at infinity. This results in richer dynamics than for the standard repulsive NLS with zero BCs at infinity—for example, traveling waves may form. On the other hand, we show that solutions with small, localized energy, disperse for large time, according to the linearized equation. Methods include certain quadratic transforms and bilinear estimates.

This is joint work with K. Nakanishi and T.-P. Tsai.

FRANCOIS HAMEL, Aix-Marseille III

Reaction-diffusion equations and transition waves

The first part of the talk is concerned with various generalizations of the usual notions of waves, fronts and propagation mean speed in a heterogeneous environment. The new notions involve uniform limits, with respect to the geodesic distance, to a family of hypersurfaces which are parametrized by time. Transition waves not only extend the standard notions to a very general setting, but, under some appropriate assumptions in classical cases, they reduce to them. General intrinsic properties, monotonicity properties and uniqueness results for almost-planar fronts have been obtained with H. Berestycki.

New geometric situations can also be covered by the new notions. As an example, we will see in a second part (with H. Berestycki and H. Matano) how to describe the propagation of a generalized almost-planar front around an obstacle for bistable reaction-diffusion equations.

YVAN MARTEL, Université de Versailles–Saint-Quentin
Collision of two solitons for nonintegrable gKdV equations

In this talk, I will report on recent results in collaboration with Frank Merle concerning the collision of two solitons for the subcritical gKdV equations $\partial_t u + \partial_x\big(\partial_x^2 u + g(u)\big) = 0$ on $\mathbf{R}$. Under general assumptions on $g(u)$, there exist stable solutions of the form $u(t,x) = Q_c(x-ct)$, called *solitons*.

Considering $0 < c_2 < c_1$, there exists a unique solution $u(t)$ such that $\lim_{t\to-\infty} \|u(t) - Q_{c_1}(.-c_1 t) - Q_{c_2}(.-c_2 t)\|_{H^1(\mathbf{R})} = 0$. Formally, the solitons have to collide. We study this collision, i.e., the behavior of the solution $u(t)$ for t bounded and for $t \to +\infty$, in the case where c_2 is small.

The first result concerns the global stability of the two soliton structure. Indeed, we prove that the two solitons survive the collision, with a possible residue and a possible change of sizes (and speeds) small with respect to Q_{c_2}. Moreover, we show monotonicity properties: the size of the large soliton does not decrease and the size of the small soliton does not increase through the collision.

Second, we focus on the quartic case $g(u) = u^4$, for which we give a sharp description of the collision. We prove that the residue is small but not zero. The collision is inelastic but very close to be elastic. This is in contrast with the integrable cases ($g(u) = u^2$ or u^3) for which the collision is elastic (zero residue).

Finally, we exhibit special symmetric solutions which extend to the nonintegrable case the notion of 2-soliton for KdV.

GREGOIRE NADIN, ENS Paris, Paris, France
Traveling fronts in space-time periodic media

This talk is concerned with the existence of traveling fronts for the equation:

$$\partial_t u - \nabla \cdot \big(A(t,x)\nabla u\big) + q(t,x)\cdot\nabla u = f(t,x,u), \tag{1}$$

where the diffusion matrix A, the advection term q and the reaction term f are periodic in t and x. We will first discuss the notion of traveling fronts in such media. Then, we will explain how to prove the existence of some speeds c^* and c^{**} such that there exists a pulsating traveling front of speed c for all $c \geq c^{**}$ and that there exists no such front of speed $c < c^*$. Lastly, in the case of a KPP-type reaction term, we will get a nice characterization of the speed c^* and give some dependence relations between the speed and the coefficients.

PIERRE RAPHAEL, Université Paul Sabatier, 31062 Toulouse, France
Rough blow up solutions to the L^2 critical NLS

The cubic nonlinear Schrödinger equation in dimension 2

$$i\partial_t u + \Delta u + u|u|^2 = 0$$

is a canonical model for the focusing of a laser beam. The question of the qualitative description of both the long time behavior of the solution and the description of the possible singularity formation has attracted a considerable amount of both formal and rigorous works for the past twenty years, in particular due to the fact that the NLS structure is considered to be canonical for a large number of problems. Following a breakthrough work of Bourgain in the 90's, a challenging problem is in particular to obtain a qualitative description of the flow in the critical space L^2. I will show how the merging of oscillatory integrals techniques as developed by Colliander, Keel, Stafilani, Takaoka and Tao for the study of the Cauchy problem for rough data, with more elliptic in nature estimates developed by Merle and Raphaël for the study of the singularity formation in the energy space H^1, allows one to derive a blow up theory in the almost critical space H^s, $\forall s > 0$.

This is joint work with Jim Colliander.

JEAN-MICHEL ROQUEJOFFRE, Université Paul Sabatier
How travelling waves attract the solutions of KPP equations

Reaction-diffusion of the KPP (Kolmogorov, Petrovskii, Piskunov) type have a semi-infinite interval of travelling wave velocities, and our goal is to examine how this family of waves attract the solutions of the evolution equation. We will mainly concentrate here on super-critical waves, i.e., those having a velocity larger than the smallest one. Well-known results (Uchiyama 1978, Bramson 1983) assert that an initial datum consisting of a super-critical wave, perturbed by a compactly—or rapidly decaying—function, will give rise to a solution converging to the initial wave, exponentially in time.

We wish to examine what happens when this assumption on the initial datum is relaxed. The scenario is the following: an initial datum trapped between two waves of the same velocity will evolve into a travelling wave profile, but with a local phase shift that may not converge to anything as time goes to infinity. In other words, convergence to a single wave does not survive.

We will describe this phenomenon, explain why it happens, and discuss some generalisations to models that are inhomogeneous in the space variable.

Joint work with M. Bages and P. Martinez.

ALEXANDER SHNIRELMAN, Department of Mathematics and Statistics, Concordia University, 1455 De Maisonneuve West, Montreal, Quebec, Canada H3G 1M8
Flow control by small forces

Consider the motion of the ideal incompressible fluid on the surface of 2-d torus $\mathbf{T}^2$ described by the Euler equations

$$\frac{\partial u}{\partial t} + (u, \nabla)u + \nabla p = 0, \quad \nabla \cdot p = 0. \tag{1}$$

Let u_0, u_1 be two stationary (time independent) solutions of (1). Consider the nonhomogeneous Euler equations

$$\frac{\partial u}{\partial t} + (u, \nabla)u + \nabla p = f, \quad \nabla \cdot p = 0, \tag{2}$$

where $f = f(x,t)$ is some external force. We say that the force f transfers u_0 into u_1 during the time T, if the solution of (2) satisfying $u(x,0) = u_0(x)$, satisfies also $u(x,T) = u_1(x)$.

Theorem *For any stationary solutions u_0, u_1 having equal energies and momenta, and for any $\varepsilon > 0$ there exist $T > 0$ and a force $f \in C^\infty(\mathbf{T} \times [0,T])$ such that f transfers u_0 into u_1 during the time T, and*

$$\max_{0 \le t \le T} \|f\|_{L^2} + \int_0^T \|f(\cdot,t)\|_{L^2}\, dt < \varepsilon. \tag{3}$$

So, the flow may be controlled by a small (in L^2) external force; most of the job is done by the fluid itself. One consequence of this fact is the absence of integrals of the Euler equations which are continuous in L^2 and different from the energy and momentum.

The proof is done by construction and uses the multiphase flow approximation for complex flows.

MICHAEL SIGAL, University of Toronto, Toronto, ON
Effective dynamics of soliton

It is a common understanding in physics that dynamics at a given scale originate from dynamics on a finer scale. In this talk I will demonstrate how this works for the case of low energy solutions of the nonlinear Schrödinger and Hartree equations with external potentials. I show how the description of such solutions can be reduced to dynamics of rigidly moving well localized structures—solitons (ground states). I will review some recent results on and state open problems in this subject.